THE PARADOX OF ADVERSARIAL LIQUIDATION IN DECENTRALISED LENDING

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ABSTRACT. We present a holistic framework for risk and reward management for decentralised lending protocols, such as AAVE or Compound. This framework highlights tensions and trade-offs in protocol design and the choice of various risks and incentives parameters.

Using this, we identify the fundamental paradox of adversarial liquidation in decentralised lending: If the rewards to liquidators are chosen such that there is an incentive for liquidators to act when a loan is in distress (as is needed for well functioning of the protocol), then it is also the case that liquidators are incentivised to manipulate prices (through front-running the liquidation process), leading to suboptimal outcomes for borrowers.

1. INTRODUCTION

Decentralised finance (Defi) protocols such as Uniswap [2], Aave [1], Compound [3] or Morpho [4], allow users to trade on the Ethereum chain without having to rely on a central trusted entity such as traditional banks [15, 13, 12]. Lending protocols in particular concentrate more than 30% of the TVL (Total Value Locked) in Ethereum's Defi protocols [1]. As a consequence there has been recent effort in the identification and mitigation of the risks associated to their components [9] [14], as well as an analysis of recent attacks associated with, but not limited to, price oracles or token's price manipulation [16] [6].

Decentralised lending protocols, such as Aave or Compound, resemble a collateralised debt market (CDM) by pooling assets from lenders to enable over-collateralised loans to borrowers. In particular, loan contracts in this market are similar to traditional finance (TradFi) stock loans and can be studied through the lens of American options, [11, 5, 7]. A novel feature of these markets, in comparison to stock loans, is that debt positions which are not sufficiently collateralised are auctioned off to liquidators at a discount. Careful design is required in order to provide the right incentives to partipants and to maintain a stable balance of protocol users under varied economic and market conditions and scenarios. The key elements in the design of lending protocols are:

- Interest rate model, which determines the cost of borrowing and yields earned by suppliers.

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[1]Data from `https://defillama.com/chain/Ethereum`

- Loan-to-value ratios at loan origination, liquidation thresholds and bonuses which control the acceptable level of risk for the protocol and its users.
- Collateralisation rules, which determine which assets can be used as collateral and control the exposure of the protocol to a particular asset class.

All the above components intrinsically rely on parameters which depend on changing market conditions and risk and reward trade-offs among market participants. Empirical evaluation of these has been done, for example, in [10, 8], but we are not aware of a study that presents a systematic analysis of risks and incentives of the current evolution of lending protocols. The main objective of this article is to fill this gap. In particular:

- We present a holistic framework for risk management for decentralised lending protocols. Our approach relies on coherent risk measures, such as Expected Shortfall, that allows idnetification of the liquidation threshold parameters such that a loan remains overcollateralised after a period of time with high probability.
- We study the properties of lending protocols, focusing on the effects and risks of liquidation, once the protocol's parameters are fixed. We explore market conditions under which liquidation spirals may happen, that is, when liquidation trigger further liquidations.
- We finally identify the fundamental paradox of adversarial liquidation in decentralised lending protocols. If the rewards to liquidators are chosen such that there is an incentive for liquidators to act when a loan is in distress (as is needed for well functioning of the protocol), then it is also the case that liquidators are incentivised to manipulate prices (through front-running the liquidation process), leading to suboptimal outcomes for borrowers.

The article is structured as follows. In Section 2 we describe in detail how lending protocols in DeFi space operate. In Section 3 we provide a high-level overview of associated risks building on what's known in TradFi. In Section 4, we analyse liquidations and spell out the paradox of adversarial liquidations. Statistical perspectives on Liquidation at Risk and collateral requirements are presented in Sections 5 and 6, respectively. We conclude by evaluating our results using simulation in Section 7.

2. DECENTRALISED LENDING PROTOCOL

2.1. **Key Variables.** For simplicity through the rest of this report we will consider the case when the loan consists of a single borrowed asset[2], which we set to be S^0 and n collateral assets $(S^1,\dots,S^n)$. Each loan position is characterised by a pair $(b_t,c_t)_{t\geq 0}$, where $b_t\in\mathbb{R}_+$ and $(c_t^1,\dots,c_t^n)\in\mathbb{R}_+^n$; these specify the amounts of borrowed and collateral assets at time $t\geq 0$, respectively. The mark-to-market value[3], of debt and collateral in a common numéraire are given by

$$b_t\cdot S_t^0 \quad \text{and} \quad \langle c_t,S_t\rangle=\sum_{i=1}^{n} c_t^i\cdot S_t^i\,.$$

[2]For the purpose of risk analysis, this means we are interested in how much collateral is required for a single debt asset. In the case of multiple borrowed assets, a collateral requirements can be aggregated, leading to a conservative approach that does not take advantage from potential netting between borrowed positions.

[3]Mark-to-market provides a convenient accounting convention, but does not take into account market frictions such as slippage, wrong-way risk, or gas and transaction fees.

It is convenient to introduce the price of collateral assets in terms of the debt asset

$$P_t^i := \frac{S_t^i}{S_t^0}.$$

With (b_t, c_t, S_t) being given, the (total) position loan-to-value (LTV) is defined as

$$\mathbb{LTV}_t := \frac{b_t \cdot S_t^0}{\langle c_t, S_t \rangle} = \frac{b_t}{\langle c_t, P_t \rangle}, \quad t \geq 0.$$

For simplicity, in what follows, we will work with the debt asset being the numéraire. In other words, we consider the borrowing being for a fixed amount of the reference asset, and the collateral assets having prices P^i.

The process of borrowing from a lending protocol at time $t_0 \geq 0$ follows the steps:

(1) Begin with supplying assets $c_{t_0} = (c_{t_0}^1, \cdots, c_{t_0}^n)$ to be used as collateral. These quantities must be nonnegative. These collateral assets earn interest (which is reinvested in the collateral asset position) according to $(I_{t_0,t}^c) = (I_{t_0,t}^{1,c}, \ldots, I_{t_0,t}^{n,c})$ so that, in the absence of a liquidation event in the time interval $[s,t]$,

$$c_t = c_s \odot I_{s,t}^c = (c_s^1 \cdot I_{s,t}^{1,c}, \ldots, c_s^1 \cdot I_{s,t}^{n,c}) \quad \text{where} \quad I_{s,t}^{i,c} := e^{\int_s^t \gamma_z^{i,c} dz}.$$

Here the (continuously compounding) interest rate for the ith collateral asset, at time t, is given by $\gamma_t^{i,c}$.

(2) The protocol's governance indicates a risk level for each collateral asset, dictated by a vector $\Theta_{\text{init}} = (\theta_{\text{init}}^1, \cdots, \theta_{\text{init}}^n)$. This has a standard interpretation as a haircut on the collateral assets' values, depending on their riskiness. Given supplied collateral c_{t_0}, its current market price P_{t_0} (with the borrowed asset being a numéraire) and Θ_{init} the limit on the amount of the asset that can be borrowed (in terms of units of the borrowed assets, as we have taken S_0 as numéraire) is

$$b_{t_0} \leq \sum_{i=1}^n \theta_{\text{init}}^i \cdot c_{t_0}^i \cdot P_{t_0}^i.$$

Here $\theta_{\text{init}}^i \cdot c_{t_0}^i \cdot P_{t_0}^i$ has an interpretation as the risk-adjusted value of the collateral provided, at time t_0, under the initial-collateral rule. We will discuss collateral rules during the lifetime of the loan below.

(3) Pay interest according to $(I_{t_0,t}^b)$ specified by the protocol's governance, in the debt asset. We assume continuous compounding with rate $(\gamma_t^b)_t$ so that, in the absence of a liquidation event in the time interval $[s,t]$ the amount of borrowed assets increases according to

$$b_t = b_s I_{s,t}^b \quad \text{where} \quad I_{s,t}^b := e^{\int_s^t \gamma_z^b dz}.$$

We observe that, provided $\theta_{\text{init}}^i \in [0,1]$ for all i, we have

$$\mathbb{LTV}_{t_0} := \frac{b_{t_0}}{\sum_{i=1}^n c_{t_0}^i \cdot P_{t_0}^i} \leq \frac{\sum_{i=1}^n \theta_{\text{init}}^i \cdot c_{t_0}^i \cdot P_{t_0}^i}{\sum_{i=1}^n c_{t_0}^i \cdot P_{t_0}^i} \leq 1,$$

that is, the loan is initially fully collateralized.

2.2. **Liquidation thresholds.** For over-collateralized loans, it is required that for all $t \geq 0$ a position satisfies $\mathbb{LTV}_t \leq 1$, that is, the value of the debt b_t is below value of the collateral (c_t, P_t). When $\mathbb{LTV}_t > 1$ a rational borrower will walk away from the loan, opening the lending protocol liquidity providers to a loss. For this reason, lending protocols allow any participant in the blockchain to repay (a percentage of) the debt for those positions with $\mathbb{LTV}_t$ close to one, in return for the corresponding percentage of the collateral plus a liquidation bonus. The bonus provides an incentive for the liquidator to take over part of delinquent parties' positions, and needs to be in line with market conditions to account for slippage, cascading liquidations, gas fees, MEV, etc.

Liquidity risk arises when no participant on a blockchain network is willing to repay a delinquent party's position. This may arise at times of high uncertainty or other market dislocations. It can fundamentally only be mitigated by collateral requirements being sufficiently large to make liquidation attractive.

In order to determine the collateral requirements during the lifetime of a loan, the collateral is valued with a haircut, similarly to the valuation at the initial time t_0. This is done using a vector $\Theta_t = (\theta_t^1, \cdots, \theta_t^n)$, with $\theta^i \in [0,1]$. The collateral is assigned a risk-adjusted value $\mathbb{C}_t = \sum_i \theta_t^i \cdot c_t^i \cdot P_t^i$, and the loan can be liquidated whenever the 'health factor' satisfies

$$\mathbb{HF}_t := \frac{\mathbb{C}_t}{b_t} \leq 1.$$

Note that

$$\mathbb{LTV}_t = \frac{b_t}{\sum_i c_t^i \cdot P_t^i} \leq \max_i \{\theta_t^i\} \frac{b_t}{\mathbb{C}_t} = \max_i \{\theta_t^i\} \frac{1}{\mathbb{HF}_t}$$

so provided $\max_i \{\theta_t^i\} < 1$ the loan is open to liquidation before $\mathbb{LTV}$ reaches 1.

Typically, in order to provide a buffer against immediate liquidation of a position (due to small moves in the values of collateral assets), the collateral values are chosen to have $\Theta_{\text{init}} = \lambda \Theta_{t_0}$, for some $\lambda < 1$. If Θ_t is a constant vector $\Theta_t = (\theta_t, \theta_t, \cdots, \theta_t)$, then $1/\mathbb{HF}_t = \frac{1}{\theta_t} \mathbb{LTV}_t$, and we are simply considering a liquidation rule based on a multiple of $\mathbb{LTV}$.

The choice of Θ must balance conflicting objectives. If Θ_{t_0} is too far from $\mathbf{1} = (1, \ldots, 1)$, a very high level of overcollateralization is demanded of borrowers. This reduces risks, but leads the protocol to be unnattractive to users. If Θ_t is too close to $\mathbf{1}$, then there is little room for a liquidation bonus, and a liquidator might not be sufficiently incentivised to close the position on time.

2.3. **Liquidations.** The percentage of debt that the liquidator can repay in a single transaction is determined by close factor parameter $\kappa_{\max} \in (0,1]$. For example, AAVE-v3 allows a liquidator to repay up to 50% debt in a single typical transaction.

The liquidation times (τ^m) are defined recursively, for $m = 1, 2, \ldots$, as

$$\tau^m = \inf\left\{ t > \tau^{m-1} : b_t \geq \mathbb{C}_t \right\}, \quad \tau^1 > t_0 .$$

It's convenient to assume that, if there is a liquidator willing to close (part of) the loan open for a liquidation, they will always do so, and therefore liquidations occur at the times $(\tau^m)_{m \geq 1}$[4].

At a liquidation event, the liquidator chooses a level $\kappa \leq \kappa_{\max}$, repays $\kappa \cdot b_{\tau^m}$ and in exchange receives $\hat{c}^i_{\tau^m}(1 + \ell^i)$ of collateral i, where $\ell^i > 0$ is the liquidation bonus in this collateral asset, and $\hat{c}_{\tau^m}$ is a solution to

$$\kappa \cdot b_{\tau^m} = \langle \hat{c}_{\tau^m}, P_{\tau^m} \rangle . \tag{1}$$

In order for this to be feasible, we clearly require that there be enough collateral held to pay the liquidator, that is, $\hat{c}^i_{\tau^m}(1 + \ell^i) \leq c^i$. Given Θ, and considering a borrower providing only a single collateral asset, this implies the condition $\theta^i(1 + \ell^i) \leq 1$ (see Lemma 1).

Here we used a vector of collateral mid prices P_{τ^m} from an external exchange that the protocol has access to via oracle (e.g chainlink). Note that equation (1) in general does not have a unique solution, and a liquidator will typically demand

[4]We could easily introduce a random delay and, for example assume, that liquidator arrives at $\tau + \delta_t$,with δ_t being a random variable calibrated to market conditions, and liquidate part of the position provided this is profitable

collateral assets that pay the highest liquidation bonus adjusted for trading costs[5]. In the absence of market frictions, a liquidator realises a profit (in the numéraire asset) of $\sum_i \ell^i \cdot \hat{c}^i_{\tau_m} \cdot P^i_{\tau_m}$.

Note that when $\mathbb{LTV}_\tau \geq 1$ it is still be possible to liquidate part of the position, but the protocol is left with a loss that needs be absorbed by an insurance fund; if that is not sufficient, the loss will be shared among liquidity providers.

3. Risks and weaknesses

We here describe, heuristically, some of the key risks facing participants in lending markets, and how the protocol can be adversarially manipulated. We will explore models for some of these risks in subsequent sections.

3.1. **Fractional reserves and bank runs.**

As is common in traditional banking, the collateral provided by borrowers is used to enable borrowing of the collateral asset; this is needed to fund the interest payments on collateral. However, this means that the collateral is not immediately available, either for liquidation or upon repayment of a loan (at which point the collateral can be withdrawn from the system), or in the event of liquidation being necessary. In this case, the system is essentially running a fractional reserve banking protocol, resulting in a risk of there being insufficient liquidity to cover short-term withdrawals from the protocol. In traditional finance, this is the well-known phenomenon of a bank-run, but in decentralized systems the traditional deposit-insurance schemes (where a central bank intervenes in the market) are not available.

As a consequence, it is important to ensure that a sufficient quantity of assets are kept liquid at any time. This can be done through limiting borrowing, and through modifying interest rates to encourage repayment of asset types which are needed by the protocol's reserves system.

3.2. **Wrong-way risk.**

Fluctuations of market prices of debt and collateral can make loan positions insolvent, that is, without rapid liquidation, it is possible for $\mathbb{LTV}_t$ to grow to exceed 1. In order to prevent this, liquidators need to be sufficiently rewarded that liquidating a position is profitable.

However, this raises the problem of wrong-way risk. Suppose the value of collateral assets (in terms of the debt asset) is falling, meaning $\mathbb{LTV}$ is rising. This is often accompanied by a loss of market liquidity for the collateral asset, making it difficult to quickly trade collateral for the debt asset in the open market. The liquidator then faces a challenge, as after liquidation they will need to exchange collateral for debt assets (as they expend debt assets during the liquidation process). Unless the liquidation reward ℓ is sufficiently large, it may not be profitable for the liquidator to act. This leads to a failure of the protocol risk management.

3.3. **Adverse selection and arbitrage.**

Ignoring transaction fees, in the situation where the market for a collateral asset is very illiquid or the value of the collateral asset is falling quickly, if the initial collateral Θ_{init} factors are not set sufficiently low, this may lead to unexpected trading patterns. For example, a trader could borrow b units of the debt asset using the lending protocol, and correspondingly deposit $c^i_t = b/(\theta^i_{\text{init}} \cdot P^i_t)$ units of a particular collateral asset. If the trader simply walks away from the lending protocol at this point (with no intention of repaying the loan), they have effectively purchased the debt asset for a unit price $(P^i_t\, \theta^i_{\text{init}})^{-1}$. If θ^i_{init} is near 1, as $(P^i)^{-1}$ is the mid price for debt asset in terms of collateral, this may be better than

[5] A rational liquidator will monitor market conditions and have an estimate of the execution costs of swapping units of collateral for the asset of his choice

the price they could achieve through trading in the market when accounting for their price impact.

3.4. **Liquidation spirals.** Given the role of liquidators, there is a potential for these systems to enter liquidation spirals, where an initial liquidation results in more liquidations becoming needed. Suppose a debt position is initially liquidated, resulting in a liquidator receiving collateral assets and expending debt assets. If the liquidator then reverses this trade on the open market, they will impact the price of the collateral asset P^i, typically decreasing it, due to their market impact. As a result, the health factor of debt positions (both of the original borrower and other borrowers using the same collateral asset) will worsen, resulting in more positions becoming open to liquidation.

3.5. **Adversarial liquidation.** As an extension of the previous issue, there is the possibility that an opportunistic liquidator will front-run the liquidation process by initially trading in the open market. Their price impact will decrease the value of collateral, depressing the health factors of borrowers, and opening up positions for liqudation. The liquidator can then liquidate their positions, for a net profit.

Whether this is possible depends on how prices in the primary market are taken into the lending protocol (through the oracle process). If the oracle provides instant accurate versions of the mid price, it seems impossible to avoid the potential for adversarial liquidation, while maintaining the profitability of liquidation. We return to this issue in the next section.

3.6. **Short Squeeze.** Another adversarial scenario, related to the liquidation risk above, is the 'short squeeze'. Suppose, as discussed, the protocol allows the collateral to be borrowed, effectively reversing the role of the collateral and debt assets. A trader could borrow a large amount of asset i and sell it on the market, pushing the price P^i down. This may trigger the liquidation of the positions that are collateralized with asset i. In turn, as described above, this can lead to liquidation spirals that further reduce P^i. At that point, the trader could a) remove a large part of the collateral used to borrow asset i in the first place (since the $\mathbb{LTV}$ for the position will decrease); b) buy back aset i at a significantly lower price than it was sold. For analysis of an attempt of the above exploit, see [6].

3.7. **Multiple protocols and arbitrage.** When there are multiple protocols, there is a competition issue surrounding collateral rules and interest payments. While it's natural to have the interest rate determined endogenously and dynamically through a competitive interaction between liquidity providers, borrowers and the protocol's reserve rules, it is not apparent that the collateral constants θ^i should be allowed to vary similarly.

In particular, any endogenization of the collateral requirements will need to be done while taking care to prevent the potential for manipulation of the collateral rules, which can form a critical point of failure for the risk management of the protocol.

4. ANALYSIS OF (ADVERSARIAL) LIQUIDATIONS

4.1. **Liquidation bonus.** When deciding how large the liquidation bonus should be, there are two primary considerations:

- The Liquidation bonus should be 'sufficiently large' so that liquidators have incentive to remove bad debt from the protocol, and borrowers do not have incentive to create bad debts. To assess what 'sufficiently large' means, it is critical to consider the market conditions that drive the execution cost of swapping redeemed collateral for the debt asset.

- The liquidation bonus should be 'sufficiently small' so that the loan position can be completely liquidated at any time, i.e. redeemed collateral together with liquidation bonus does not exceed available collateral. This essentially corresponds to a choice of Θ_t, which needs to be sufficiently small to manage risk, without overly disincentivizing borrowers from using the protocol.

The role of the liquidator is similar to that of an arbitrageur in financial markets. A stylized liquidator operates as follows:

i) Begin with zero capital[6].
ii) Choose a level $\kappa < \kappa_{\max}$ of debt to liquidate. Borrow $\kappa \cdot b_{\tau^m}$ (e.g using a flashloan) of debt assets.
iii) Liquidate (part of) the debt position, by paying $\kappa \cdot b_{\tau^m}$ in debt assets to receive the collateral vector[7] with liquidation bonus $\hat{c}_{\tau^m} \odot (1+\ell)$, where $\hat{c}_{\tau^m}$ is determined by (1)
iv) At the external exchange (assuming no market friction), swap the vector $\hat{c}_{\tau^m}$ of collateral assets for debt assets, to obtain $\kappa \cdot b_{\tau^m}$ (as determined by (1)).
v) Pay back the initial loan of $\kappa \cdot b_{\tau^m}$ and keep the profit vector $\hat{c}_{\tau^m} \odot \ell$.

In practice step (4) is unrealistic as there are necessary execution costs due to slippage, MEV, gas fees, etc. Denote by $\Delta_t(x)$ the price slippage (inclusive of MEV, gas fee, etc.) when trading an amount x on the market [8]. That is, the amount received (in terms of the debt asset) when selling a vector x of collateral assets is given by $\langle x, P_t - \Delta_t(x) \rangle$.

With this notation, in order to obtain $\kappa \cdot b_{\tau^m} = \langle \hat{c}_{\tau_m}, P_{\tau_m} \rangle$ in debt assets on the open market, the liquidator will need to solve for x in the equation

$$\langle x, P_{\tau^m} - \Delta_{\tau^m}(x) \rangle = \langle \hat{c}_{\tau_m}, P_{\tau_m} \rangle . \tag{2}$$

In order for this to be profitable, there needs to be a solution with $x \le \hat{c}_{\tau_m} \odot (1+\ell)$ componentwise, at least for some $\hat{c}_{\tau_m}$ with $\kappa \cdot b_{\tau_m} = \langle \hat{c}_{\tau_m}, P_{\tau_m} \rangle$ and $\hat{c}_{\tau_m} \le c_{\tau_m}$. In turn, this forces a relationship between ℓ and the slippage Δ.

Furthermore, a sophisticated liquidator might also decide to unwind $\hat{c}_{\tau^m}$ over some period of time $[\tau^m, \tau^m + T]$, for a horizon $T > 0$, to balance potentially significant execution costs (especially during thin liquidity periods) with adverse price movements. However, it is important to note that this exposes them to heightened wrong-way risk, as this typically needs to be done in a situation where the relative value of collateral P_{τ_m} is falling.

In the case of multiple collateral assets, it is typically permitted for a liquidator to redeem one collateral asset at a time. This is convenient, as it allows us to reduce our analysis accordingly.

Our first observation, in the following lemma, considers whether there is sufficient collateral to liquidate a position, accounting for the liquidation bonus.

Lemma 1 (Ability to liquidate fully for one-collateral loan)**.** *Consider a loan with liquidation bonus $\ell^i \in \mathbb{R}_{>0}$. There is sufficient collateral to liquidate the position fully, accounting for the liquidation bonus, if and only if*

$$\sum_i \frac{c^i_{\tau_m} P^i_{\tau_m}}{1+\ell^i} \le b_{\tau_m} .$$

[6]This is simply to avoid having to account for the interest they should earn on their initial capital

[7]We recall that we write $\odot$ for componentwise multiplication of two vectors, and represent portfolios of collateral assets by vectors.

[8]This could be estimated from data e.g one could assume polynomial slippage and estimate the order and coefficients of the polynomial by running regression analysis, similarly one could run a predictive model for gas fee and MEV

Assuming any combination of collateral assets can be held, this yields the condition

$$\theta^i(1+\ell^i) \leq 1\,,$$

relating the protocol thresholds and liquidation bonuses.

Proof. We omit the subscript τ^m for notational simplicity. Consider the extreme case where all collateral assets are given to a liquidator, with no remainder. That is, the liquidator will pay the full value b of the debt asset, and receive in payment the vector $c = \hat{c} \odot (1+\ell)$. For this to correspond to the cost of repaying the asset, we must have

$$b = \langle \hat{c}, P \rangle = \sum_{i=1}^{n} \frac{c_i}{1+\ell^i} P^i.$$

Considering a position collateralized by a single asset, we have

$$b = \frac{c_i}{1+\ell^i} P^i.$$

However, at the first moment this position can be liquidated, we also know $b = \theta_t^i c_i P^i$, so rearrangement gives $\theta_t^i = 1/(1+\ell^i)$. This gives the binding constraint between θ^i and ℓ^i, and clearly reducing either θ^i or ℓ^i will lead to a situation where there is (more than) sufficient collateral to liquidate the position fully.

□

Following partial liquidation, it is not immediately clear that the health ratio will have improved sufficiently, in particular it may be subject immediately to a further liquidation. This is because liquidation both reduces the debt (improving the health ratio), and extracts a liquidation bonus (worsening the health ratio). The following lemma provides the necessary condition to ensure that liquidation improves the health ratio overall.

Lemma 2. *At a liquidation time τ_m, the health factor $\mathbb{HF}$ will increase (i.e. improve) for all liquidation strategies if and only if the health factor before liquidation satisfies*

$$\mathbb{HF}_{\tau_m} > \max_i \{\theta^i(1+\ell^i)\}.$$

In particular, if $\max_i\{\theta^i(1+\ell^i)\} < 1$ *(as suggested by Lemma 1) the health factor will always improve, provided liquidation occurs sufficiently quickly after a position becomes open for liquidation, and prices are continuous.*

Proof. To begin with, suppose the liquidator chooses to receive payment in the jth collateral asset. That is, they will repay $\kappa \cdot b_{\tau^m}$ of the debt, and receive $\hat{c}^j(1+\ell^j)$ amount of collateral asset j, where $\hat{c}^j \in [0, c^j_{\tau^m}]$ solves

$$\kappa \cdot b_{\tau^m} = \hat{c}^j \cdot P^j_{\tau^m}\,.$$

Before liquidation, the health factor is given by

$$\mathbb{HF}_{\tau^m} = \frac{\mathbb{C}_{\tau^m}}{b_{\tau^m}} = \frac{\sum_i \theta_t^i \cdot c_t^i \cdot P_t^i}{b_t}.$$

Immediately after liquidation, the health factor becomes

$$\begin{aligned}\mathbb{HF}_{\tau^m+} = \frac{\mathbb{C}_{\tau^m+}}{b_{\tau^m+}} &= \frac{\theta_t^j \cdot (c^i_{\tau^m} - \hat{c}^j(1+\ell^j)) \cdot P^i_{\tau^m} + \sum_{i\neq j} \theta^i_{\tau^m} \cdot c^i_{\tau^m} \cdot P^i_{\tau^m}}{(1-\kappa) b_{\tau^m}} \\ &= \frac{1}{1-\kappa}\mathbb{HF}_{\tau^m} - \frac{\theta^j(1+\ell^j)\kappa}{1-\kappa}.\end{aligned}$$

Rearranging, we have

$$\mathbb{HF}_{\tau^m+} - \mathbb{HF}_{\tau^m} = \frac{\kappa}{1-\kappa}\Big(\mathbb{HF}_{\tau^m} - \theta^j(1+\ell^j)\Big) \tag{3}$$

Therefore, in order for the health factor to have improved, this equation must be positive, in other words, as $\kappa \in (0,1)$,

$$\mathbb{HF}_{\tau^m} > \theta^j(1+\ell^j).$$

For more general liquidations, we can consider the liquidation taking place sequentially in each collateral asset. The result follows.

□

Remark 1. *Note that the conditions stated in Lemmas 2 and 1 do not depend on the close factor κ. This implies that the choice of κ is dictated by: a) how much the protocol penalises borrowers for triggering the liquidation threshold; b) the amount of collateral needed to liquidate delinquent loan positions. This is linked to Liquidation-at-Risk, which assesses how much of each type of collateral the protocol needs to ensure that required liquidations are possible with sufficiently high probability.*

4.2. **Liquidation Spirals.** To extend the analysis of the previous section, to account for how liquidations affect prices, we need to build a model for the price impact of our stylized liquidator's trades.

For simplicity, we will assume that there is a single collateral asset. We recall that we have already defined the effect of slippage to be given by Δ_t, that is, trading $x > 0$ collateral assets for debt assets, the price realized by the liquidator will be $P_t - \Delta(P_t, x)$. We have augmented our earlier definition slightly, to allow Δ to depend on the current price P. Similarly, we define a function H, which describes the impact on the price of collateral assets caused by this trade – after the trade, the new price will be given by $P_t - H(P_t, x)$. Basic economics of supply and demand suggest that $\Delta > 0$ and $H > 0$. In most cases Δ and H cannot be known perfectly in advance, but can be estimated from historical data. One setting where perfect knowledge is possible is the setting of an automatic market maker, as shown in the following example.

Example 1 (Uniswap v2)**.** *Suppose the open market is described by a constant product market (CPM) (e.g. Uniswap v2), with fixed gas fees g. The CPM pool contains debt assets and collateral assets, with the current relative price P_t. If X_t denotes the quantity of collateral assets in the pool, and Y_t the quantity of debt assets, then a constant product market is specified by $X_tY_t = v$ for some $v > 0$. The current price indicates that X_t and Y_t should initially be in the ratio $Y_t/X_t = P_t$, which implies $Y_t = \sqrt{vP_t}$, and $X_t = \sqrt{v/P_t}$.*

A trader can exchange x collateral assets for y debt assets provided $(X_t + x)(Y_t - y) = v$. We can now calculate our slippage and impact functions, based on the fact that the trade will satisfy

$$y = Y_t - \frac{v}{X_t + x} = \sqrt{vP_t} - \frac{v}{\sqrt{v/P_t} + x} = \frac{P_t x}{x\sqrt{P_t/v} + 1}.$$

Using this, we obtain, for $x \neq 0$,

$$\Delta(P_t, x) = P_t - \frac{y}{x} + \frac{g}{x} = P_t - \frac{P_t}{x\sqrt{P_t/v} + 1} + \frac{g}{x},$$

$$H(P_t, x) = P_t - \frac{Y_t - y}{X_t + x} = P_t - \frac{\sqrt{vP_t} - (P_t x)/(x\sqrt{P_t/v} + 1)}{\sqrt{v/P_t} + x} = P_t - \frac{P_t v}{(x\sqrt{P_t} + \sqrt{v})^2}.$$

We scale the gas fee as it does not dependent on the trade size.

Example 2 (CFM)**.** *Consider a general Constant Function Market (CFM) characterised by a bonding function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}$ which determines the state of the pool after each trade according to the acceptable fund positions:*

$$\left\{(x,y) \in \mathbb{R}_+^2 \,:\, \Psi(x,y) = constant\right\}.$$

Let k be the depth of the CFM pool. By the implicit function theorem, there exists a convex function $\psi : \mathbb{R} \to \mathbb{R}$*, called the* trading function *(or level function), s.t.*

$$\Psi(x,y) = v \qquad \Leftrightarrow \qquad \psi(x) = y\,.$$

As in the previous example, let X_t *denote the quantity of collateral, and* Y_t *the quantity of debt assets in the pool, respectively. The volume-weighted average price for x, denoted* $P_t(x)$*, is given by*

$$Y_t - y = \psi(X_t + x) \implies P_t(x) := \frac{y}{x} = -\frac{\psi(X_t + x) - \psi(X_t)}{x}\,.$$

In particular we see that $P_t = \lim_{x\to 0+} P_t(x) = -\psi'(X_t)$*. Hence the execution cost and permanent impact functions for the trade of size x are given by*

$$\begin{aligned}\Delta(P_t,x) &= P_t - P_t(x) + \frac{g}{x} = \frac{\psi(X_t+x) - \psi(X_t)}{x} - \psi'(X_t) + \frac{g}{x}\\ H(P_t,x) &= +\psi'(X_t + x) - \psi'(X_t)\,.\end{aligned}$$

We scale the gas fee as it does not dependent on the trade size.

Given these functions, we can return to our stylised liquidator who, after liquidating a position, finds themselves short κb units of the debt asset and long $(1+\ell)\kappa b/P_t$ units of collateral. We suppose that (given fees and price impact) they will trade a minimal quantity on the open market to offset their short position, that is, they will buy κb units of debt assets, for which they pay x units of collateral, where $\kappa b = x(P_t - \Delta(P_t,x))$. Provided

$$x < \frac{(1+\ell)\kappa b}{P_t} \qquad \Leftrightarrow \qquad 1 < (1+\ell)\Big(1 - \frac{\Delta(P_t,x)}{P_t}\Big), \tag{4}$$

this trade is profitable for the liquidator. Comparing with our criterion $\theta(1+\ell) < 1$ (which is needed to ensure the liquidation rewards can be paid out of the collateral), we see that we need to ensure

$$\theta < \frac{1}{1+\ell} < 1 - \frac{\Delta(P_t,x)}{P_t}$$

in order for the liquidation procedure to be effective. We formalise this observation as follows:

Lemma 3. *Assume the slippage for trading x amount of collateral on the external market with mid price* P_t *is given by* $\Delta(P_t,x)$*. Liquidation is profitable for the liquidator and there is sufficient capital to liquidate the position fully if and only if the liquidation bonus* ℓ *satisies*

$$1+\ell \in \Big[\frac{P_t}{P_t - \Delta(P_t,x)}, \frac{1}{\theta}\Big].$$

This further emphasises the importance of the liquidation reward ℓ and threshold θ depending on market conditions, and provides additional guidance on the appropriate choice of θ, when compared with the statistical approach which we present in Section 6.

Once the liquidator has made these trades on the open market, the price will have moved to $P_t - H(P_t,x)$. As $H > 0$ (typically), we see that this is a decrease in the value of the collateral asset. This has the effect of decreasing the health factor of loans, as the collateral is worth less. In turn this may lead to risk that we will enter a single-portfolio liquidation spiral, where the health factor is decreased enough to enable liquidation to reoccur immediately. In the next lemma we provide necessary and sufficient conditions for this to happen.

Lemma 4 (Single-asset liquidation spiral)**.** *Liquidation of a position causes further liquidations of the same position if and only if*

$$\left[1+\frac{\kappa}{1-\kappa}\left(1-\theta^j(1+\ell^j)\right)\right]\left(1-\frac{H(P_t,x)}{P_t}\right)<1,$$

where x solves $\kappa b = x(P_t - \Delta(P_t,x))$.

Proof. We write $\mathbb{HF}_{t-}$ for the health factor before (partial) liquidation, $\mathbb{HF}_t$ after liquidation, and $\mathbb{HF}_+$ after the liquidators has offset their position in the market. We know

$$\mathbb{HF}_{t+} = \frac{\theta c_t(P_t - H(P_t,x))}{b_t} = \mathbb{HF}_t\left(1-\frac{H(P_t,x)}{P_t}\right).$$

We now compare with Lemma 2, to ask whether this impact is sufficient to trigger further liquidations *in the same position*. Compared with $\mathbb{HF}_{t-}$ (the health factor before liquidation), cf. equation (3), we have

$$\begin{aligned}\mathbb{HF}_{t+} &= \mathbb{HF}_t\left(1-\frac{H(P_t,x)}{P_t}\right)\\ &= \mathbb{HF}_{t-}\left[1+\frac{\kappa}{1-\kappa}\left(1-\frac{\theta^j(1+\ell^j)}{\mathbb{HF}_{t-}}\right)\right]\left(1-\frac{H(P_t,x)}{P_t}\right)\end{aligned}$$

If we assume liquidation occurs at the first possible moment, so $\mathbb{HF}_{t-}=1$, then

$$\mathbb{HF}_{t+} = \left[1+\frac{\kappa}{1-\kappa}\left(1-\theta^j(1+\ell^j)\right)\right]\left(1-\frac{H(P_t,x)}{P_t}\right). \tag{5}$$

The conclusion follows. □

4.3. **Multi-loan spirals and adversarial liquidation.** There is a second possible form of liquidation spiral, where a position which holds sufficient collateral finds its health factor decreased *due to the liquidator's actions on another loan*. From the perspective of this borrower, this is no different to the situation where a liquidator acts preemptively in the market, in order to depress the value of collateral and thereby profit.

Definition 1 (Health factor at risk at level x)**.** *We say that a loan's health factor is at risk at level x if*

$$\mathbb{HF}_t \le \left(1-\frac{H(P_t,x)}{P_t}\right)^{-1}.$$

A loan whose health factor is at risk at level x will be open to liquidation whenever a trade of size x is made in the market.

Suppose the liquidator chooses to front-run the liquidation process, that is, they trade in the open market first. If there is a loan whose health factor is at risk at level x, then the liquidator is able to offset their trade though liquidating a loan. The sequence of events, in this case, is as follows:

(i) A liquidator notices that there is a loan whose health factor is at risk at level x, where x solves
$$x = \frac{\kappa b}{P_t - \Delta(P_t,x)}, \tag{6}$$
for b the value of the loan and κ the liquidation fraction.

(ii) Liquidator exchanges x units of collateral for $\kappa b = x[P_t - \Delta(P_t,x)]$ debt assets. This causes the price of the collateral to decrease to $P_t - H(P_t,x)$.

(iii) The health factor of a loan is affected by the price change, decreasing from $\mathbb{HF}_t$ to $\mathbb{HF}_t \times \frac{P_t - H(P_t,x)}{P_t} \le 1$.

(iv) The liquidator (partly) liquidates the loan, exchanging their κb debt assets for $(1+\ell)\frac{\kappa b}{P_t - H(P_t,x)}$ collateral assets. Their net profit (in units of collateral assets) is

$$(1+\ell)\frac{\kappa b}{P_t - H(P_t,x)} - \frac{\kappa b}{P_t - \Delta(P_t,x)} = x(1+\ell)\frac{P_t - \Delta(P_t,x)}{P_t - H(P_t,x)} - x\,.$$

Therefore, this is a profitable trade whenever

$$1+\ell > \frac{P_t - H(P_t,x)}{P_t - \Delta(P_t,x)}. \tag{7}$$

In the previous section (cf. (4)), we argued that, in order for the liquidation process to be effective (i.e. in order to ensure profitability of liquidating loans with health factors below 1), we require the reward ℓ to satisfy

$$1+\ell > \frac{p}{p - \Delta(p,x')}$$

where x' satisfies $\kappa b = x'(p - \Delta(p,x'))$ and p is the current price. After the front running trade (step (ii) above), the price has decreased to $P_t - H(P_t,x)$, which implies, if the oracle price is immediately updated, we require

$$1+\ell > \frac{P_t - H(P_t,x)}{(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')}, \tag{8}$$

where

$$\kappa b = x'\left((P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')\right).$$

In the next theorem we compare right hand sides of (8) and (7) for the generic CFM market.

Theorem 1. *Consider the CFM with level function $\Psi(x,y) = k$, and convex trading function $y = \psi(x)$. Then we have*

$$\frac{P_t - H(P_t,x)}{(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')} > \frac{P_t - H(P_t,x)}{P_t - \Delta(P_t,x)},$$

where x,x' are the consecutive trades that buy κb, solving

$$\kappa b = x\,(P_t - \Delta(P_t,x))$$
$$\kappa b = x'\left((P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')\right).$$

The proof is provided in Appendix A . This theorem tells us that if the slippage and the price impact are derived from a CFM with convex trading function, we are faced with a conundrum, as

$$1+\ell > \frac{P_t - H(P_t,x)}{(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')} > \frac{P_t - H(P_t,x)}{P_t - \Delta(P_t,x)}.$$

Therefore, if (8) holds, (7) also holds. This leads to a fundamental paradox of adversarial liquidation protocols.

> *If the rewards to liquidators are chosen such that there is an incentive for liquidators to act when a loan is in distress (as is needed for well functioning of the protocol), then it is also the case that liquidators are incentivised to manipulate prices (through front-running the liquidation process), leading to suboptimal outcomes for borrowers.*

We can also see from this that there is an implied bound on the health factor, which determines whether adversarial liquidation is possible. This is given by

$$\mathbb{HF}_t \leq \left(1 - \frac{H(P_t, X(b,P_t))}{P_t}\right)^{-1}$$

where $X(b,p)$ is defined as the maximal solution to $x = \kappa b/(p-\Delta(p,x))$ (cf. equation (6)). Any loan with a health factor below this bound can be profitably liquidated with appropriate frontrunning trades. In the context of a Uniswap v2 market, these can be explicitly calculated, however the formulae do not simplify elegantly.

5. Liquidation at risk

From the perspective of the protocol, one concern is that the capital (which needs to be given to the liquidator, in the event of liquidation), is typically rehypothecated to allow borrowing. It is therefore important to determine how much capital should be kept on hand to enable efficient liquidations. From the above calculation, we see that it is important not only to consider the number of loans whose health factor is low, but also those whose health factor could be at risk given price manipulation on the part of the liquidator.

Suppose the protocol has a set of loans with current states $(b^j, c^j)_{j\geq 1}$, where b^j records the (current) amount of the loan and c^j records the (current) amount of collateral assets. We assume that the protocol knows (or can accurately estimate), the price impact functions Δ and H, and so can compute the function $X(b,p)$ defined above.

We first determine how many loans can be liquidated if there is an external price fall to p. Given the health factor requirement, at price p, a loan can be liquidated if $\theta c^j p \leq b$. From the perspective of the protocol, this implies a *first-order liquidation-requirement*, at price p, in terms of units of collateral, is given by $\mathscr{L}_1(p)$, for

$$\mathscr{L}(p) = (1+\ell)\sum_j \frac{\kappa b^j}{p}\mathbf{1}_{\{\theta c^j p \leq b^j\}} = \frac{\kappa(1+\ell)}{p}\sum_j b^j \mathbf{1}_{\{\theta c^j p \leq b^j\}},$$

where θ is the haircut risk coefficient appearing in the (single collateral) health factor. This is the total amount of collateral needed for (part) liquidation of all positions which need to be liquidated at price p, and $\mathscr{L}(p)$ is the total size of the position liquidated, in terms of units of collateral (including the liquidation reward).

However, we have seen above that this liquidation may involve liquidators acting in the open market, to reduce the value of collateral – either through front-running the liquidation process, or through offsetting their resulting collateral positions. Assuming liquidators will liquidate all loans which is is profitable to do so, we describe what occurs when the price exogenously moves to p, and front-running can occur. We denote by $\mathscr{X}(p)$ the total quantity of collateral which needs to be traded for debt assets (in step (ii) of the process, by the front-running liquidator);

We assume the liquidator will optimize their strategy, that is, will maximize the final number of collateral assets they hold at the end of the process. This yields the implicit equation for $\mathscr{X}(p)$:

$$\mathscr{X}(p) = \arg\max_x \Big\{\mathscr{L}(p - H(p,x)) - x\Big\}$$

It is clear that $\mathscr{L}$ is decreasing in p, so if H is increasing in x (as is usually the case), we know that the optimal trade size is given by a trade-off between the two terms in the maximization. We can also see that $x \mapsto \mathscr{L}(p-H(p,x)) - x$ is typically decreasing in x, but can jump upwards whenever $p - H(p,x)$ crosses one of the values $b^j/(\theta c^j)$. Therefore, the maximizer must occur at a point $x \in \{0\} \cup \{b^j/(\theta c^j)\}_{j>0}$. This results in a discrete search problem, which can be efficiently solved offline, and a variety of approximations can be considered (for example, if the number of loans is very large, a smooth approximation of the sum in $\mathscr{L}_1$ can be considered).

Given a potential future price p, it is then possible to determine the required treasury holdings of the protocol by computing

$$[-\mathscr{L}(p - H(p, \mathscr{X}(p)))] \,.$$

If p is taken to be random, statistics such as the expected shortfall (discussed in the next section) can be computed.

Remark 2. *The computations presented here can easily be extended to a setting with multiple collateral assets, provided we can keep track of price impacts between all the assets.*

6. Collateral requirement

The initial collateral requirements, as represented through the vector Θ_{init}, dictate how much of the debt asset can be borrowed at time zero given posted collateral. As market prices evolve, it may be that the value of collateral posted (as always, using the debt asset as a numéraire) falls, decreasing the health factor of the loan. A collateral requirement is an ongoing restriction on the health of the position which is acceptable within the market and should account for the risk to lenders of the possibility of default.

The requirement can be computed as follows: Fix a period of time $h > 0$ e.g. 30 seconds, 1 hour or 24 hours. This 'liquidation horizon' must be sufficiently long for all participants in the market to be able to observe the state of the loan and execute transactions on the blockchain.

The protocol's governance needs to choose a threshold Θ based on current market conditions. This can be motivated by looking at the risk to a liquidator (or to liquidity providers, if no external liquidator steps in), over the stated horizon. In order to be consistent, this should also account for the interest payments which fall due within this horizon.

To evaluate this risk, we fix a small $\alpha \in [0,1)$ and for any time $t \geq 0$ require b_t to satisfy[9]

$$\underbrace{\mathbb{P}\Big(\underbrace{\langle c_{t+h}, P_{t+h}\rangle}_{\text{collateral value}} - \underbrace{b_t e^{\int_t^{t+h}\gamma_z dz}}_{\text{debt with interest}} \leq 0\Big)}_{\text{probability of loss}} \leq \alpha \,.$$

In other words, given initial collateral c_t the amount b_t should be chosen so that at fixed time $h > 0$ the loan position remains over-collateralized with high probability $(1-\alpha)$. This would lead us to set the the maximum amount which can be borrowed to

$$\sup\Big\{b \in \mathbb{R} : \mathbb{P}\Big(e^{-\int_t^{t+h}\gamma_z dz}\langle c_{t+h}, P_{t+h}\rangle - b \leq 0\Big) \leq \alpha\Big\} \,.$$

We observe that this is simply the negative of the Value-at-Risk of the discounted collateral

$$\mathrm{VaR}_\alpha((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle) = \inf\Big\{b' \in \mathbb{R} : \mathbb{P}\Big((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle + b' \leq 0\Big) \leq \alpha\Big\} \,,$$

where recall that $I^b_{t,t+h} = e^{\int_t^{t+h}\gamma^b_z dz}$.

[9] An alternative approach would be to set b_t so that

$$\mathbb{P}\left(\inf_{t' \in [t,t+h]}\left\{e^{-\int_t^{t+t'}\gamma_z dz}(c_{t'}, P_{t'}) - b_t\right\} \leq 0\right) \leq \alpha \,.$$

The expected shortfall computation in that case would be more demanding and we expect impact of liquidations and liquidation spirals in particular, which we discuss in the next section, to be more significant.

Here $b = -b'$ will be positive, and is interpreted as the amount of debt asset that can be borrowed based on the collateral. However, $\mathrm{VaR}_\alpha((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle)$ does not tell us the size of the losses in case the collateral c is insufficient. For this reason it makes more sense to set the initial borrowed amount at the negative of the expected shortfall in case the VaR is insufficient:

$$\begin{aligned}\mathrm{ES}_\lambda((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle) &= \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle)\,d\alpha \\ &= \mathbb{E}\Big[-(I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle \Big| \langle c_{t+h}, P_{t+h}\rangle < -\mathrm{VaR}_\lambda((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle)\Big].\end{aligned} \tag{9}$$

The last equality tells that the negative of the expected shortfall is the average value of the collateral, conditioned on the event that value of the collateral falls below the level $-\mathrm{VaR}_\alpha((I^b_{t,t+h})^{-1}\langle c_{t+h}, P_{t+h}\rangle)$.

To be able to compute the expected shortfall (9) one needs to model the joint behaviour of all collateral assets and approximate the conditional expectation which in general is computationally demanding, statistically uncertain, and often requires combination of parametric approximations with Monte Carlo simulations. Particularly in a blockchain setting, this is computationally impractical.

For this reason, it is common to make the simplifying assumption that no offsetting is permitted among the collateral assets. In addition, for the sake of robustness, we can also assume that the interest earned on collateral during the liquidation horizon is not included. In this case, we can consider using only the jth collateral asset, and derive the condition (using properties of the expected shortfall)

$$b \le \mathrm{ES}_\lambda(\langle c_t, P_{t+h}\rangle) = \mathrm{ES}_\lambda(c^j_t \cdot P^j_{t+h}) = c^j_t \cdot \frac{\mathrm{ES}_\lambda(P^j_{t+h})}{P^j_t} \cdot P^j_t. \tag{10}$$

By comparison with the health factor defined above, this suggests the choice of risk factors

$$\theta^j_t = \frac{\mathrm{ES}_\lambda(P^j_{t+h})}{P^j_t} = \mathrm{ES}_\lambda\Big(\frac{P^j_{t+h}}{P^j_t}\Big).$$

That is, the risk factor θ^j should be set as the expected shortfall of the return of the collateral asset (in terms of the debt asset as numéraire) over the desired liquidation horizon.

Even in this simplified setting, we still require a model for the price returns of the collateral assets. This involves statistical modelling, which should account for wrong-way risk (that liquidations will typically occur in periods of heightened market stress), leading to exaggerated estimates of the expected shortfall, when compared with usual market conditions. This modelling, which results in the estimation of risk factors θ^j, will typically need to occur off chain.

We have also not directly incorporated the price impact of potential liquidations that may happen over $[t, t+h]$ (which we describe in more detail in the next section).

7. Numerical simulations

Throughout this section we focus on the pair WETH-USDT, where USDT is the borrowed asset and WETH is the collateral asset. We use the Uniswap v2 pool for this pair as the reference market. We refer the reader to Appendix B for more details about the data that we use in our experiments.

We divide this section in

(1) A simulation subsection that requires a model calibration for the price of WETH-USDT to historical data, in order to calculate (a) the liquidation threshold θ and (b) the collateral at risk, under different modelling assumptions.

(2) A subsection showing the properties of of over-collateralized lending protocols under different liquidity scenarios of the underlying market. In this subsection it is not so important to show how the liquidation threshold θ is derived under some modelling assumption, but to provide a visualisation of the possible effects of liquidation once θ and the liquidation bonus ℓ are fixed.

7.1. **WETH-USDT model calibration.** Consider an open position with USDT as the borrowed asset and WETH as the only deposited collateral asset. We set t as the 1st of January of 2023, and fix h to be 12 days after t[10]. The calculation of $\mathrm{ES}_\lambda(P_{t+h})$ in (10) requires some modelling assumption on P_t. We model it by a SABR model

$$\begin{aligned} dP_t &= \mu_t dt + V_t (P_t)^\beta dW_t^1, \quad && P_0 = p_0 \\ dV_t &= \nu \cdot V_t dW_t^2, && V_0 = \alpha_0 \\ d\langle W_t^1, W_t^2 \rangle &= \rho dt \end{aligned} \tag{11}$$

where $W = (W_t^1, W_t^2)$ is a 2-dimensional Brownian motion. We calibrate the parameters $\beta, \alpha_0, \nu, \rho$ to call options prices data, take $p_0 = 1\,195.7$ the price of ETH in USD on the 01/01/2023, and set the drift so that we get three different scenarios,

- μ_t is such that $(\mathbb{E}(P_t))_{t \geq t_0}$ is monotone decreasing,
- $\mu_t = 0$,
- μ_t is such that $(\mathbb{E}(P_t))_{t \geq t_0}$ is monotone increasing.

Figure 1 provides sample paths for the calibrated models.

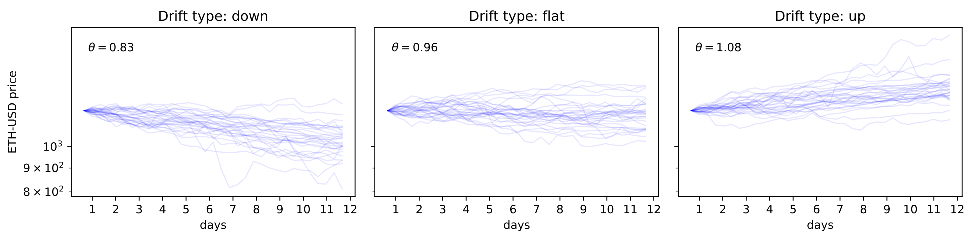

FIGURE 1. Liquidation threshold θ for three drift scenarios.

7.1.1. *Liquidation threshold θ.* We use Monte Carlo simulations of (11) to approximate the risk factor $\theta = -\mathrm{ES}_\lambda \left(\frac{P_{t+h}^j}{P_t^j} \right)$ where $h = 12$ days, see Figure 1. For each of the above three scenarios, we get $\theta = 0.83, \theta = 0.96, \theta = 1.08$ respectively.

Note that if μ_t is calibrated so that $\mathbb{E}(P_t)$ increases in time, then on average the predicted loan-to-value decreases, and under this model it would be enough for a borrower to undercollateralize their loan (since $\theta > 1$) if there was a mechanism to stop them from walking away. However, in this case, the bound on the liquidation bonus established in Lemma 3 is violated.

7.1.2. *Liquidation at risk.* We derive the liquidation at risk

$$-\mathrm{ES}_\alpha[-\mathscr{L}(P_{t+h} - H(P_{t+h}, \mathscr{X}(P_{t+h})))],$$

under the three modeling assumptions for the drift μ_t for the price P_t of ETH in USD, and under liquidity scenarios considered in Table 1, namely

[10]The choice of h is arbitrary and corresponds to protocol governance risk appetite

– Liquidity scenario queried from historical data on using GraphQL on the 1st of January of 2023. The pool liquidity constant is $k = 108544044156.00478$, with initial price $P_t = 1195.16$.
– Low liquidity scenario. Liquidity is $10^{-4}k$ where k is the real liquidity, and we fix the initial price of WETH in USD $P_t = 1195.16$
– Very low liquidity scenario. Liquidity is $10^{-5}k$ where k is the real liquidity, and we fix the initial price of WETH in USD $P_t = 1195.16$.

We consider a range for the risk parameter $\theta \in [0.7, 1)$ and we also consider two open positions with initial collateralization scaling 0.8 and 0.9:

(1) A very overcollateralized position with

$$c_t = 1 \text{ WETH}, \quad b_t = c_t \cdot P_t \cdot \theta \cdot 0.8 \text{ USDT}$$

for each θ that we consider in the range $[0.7, 1)$.

(2) A slightly less overcollateralized position with

$$c_t = 1 \text{ WETH}, \quad b_t = c_t \cdot P_t \cdot \theta \cdot 0.9 \text{ USDT}$$

for each θ that we consider in the range $[0.7, 1)$.

Using the calibrated SABR model for P_t we calculate the liquidity at risk $-(1+\ell)\mathrm{ES}_\alpha[\mathscr{L}(P_{t+h} - H(P_{t+h}, \mathscr{X}(P_{t+h})))]$ via Monte Carlo simulations. That is, for each Monte Carlo sample $p_{t+h} \sim P_{t+h}$ to find $\mathscr{L}(p_{t+h} - H(p_{t+h}, \mathscr{X}(p_{t+h})))$, which we can use to calculte the liquidity at risk as a Monte Carlo average.

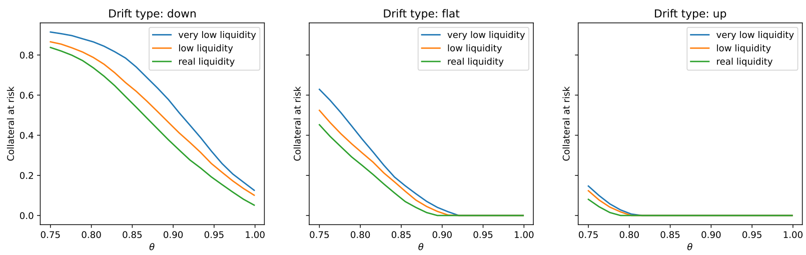

FIGURE 2. Liquidation at risk

We see in Figure 2 that lower values of θ yield a higher collateral at risk, since adversarial liquidations can be triggered for lower values of P_{t+h}. If we model P_t with a drift such that $\mathbb{E}(P_t)$ is decreasing in time, then adversarial liquidations will be triggered with higher probability at $t+h$, yielding higher collateral at risk values in all the θ regimes and all the liquidity regimes. Finally, lower liquidity implies a higher slippage $\Delta(p,x)$ which, as we have seen in previous examples, also encourages adversarial liquidations.

7.2. **Properties of overcollateralized protocols.** Throughout this section, we consider the three liquidity scenarios from Table 1. We consider USDT as the borrowed asset, and WETH as the collateral asset, and $P_t = 1195$ the price of WETH in USD for $t = 01/01/2023$. We will fix the borrowed amount to be $b = 5000$ USDT, $\theta \in [0.75, 1)$, $c = \frac{b}{P_t \cdot \theta \cdot 0.95}$ WETH (that is, the loan is overcollateralised above the liquidation threshold). The liquidation price at which $\mathbb{HF}_t = 1$ satisfies $P_{\text{liq}} := \frac{b}{c \cdot \theta}$.

7.2.1. *Liquidation bonus bounds.* We study the bounds for $(1+\ell)$ so that the liquidation procedure is profitable for the liquidator, under different liquidity scenarios.

We consider Uniswap v2 as the referece market, where $\Delta(P_t, x)$ and $H(P_t, x)$ have explicit form provided in Example 1. For simplicity, we assume zero-gas fees.

We have stated that the bonus bounds for liquidation to be effective are

$$1+\ell \in \left[\frac{P_{\text{liq}}}{P_{\text{liq}} - \Delta(P_{\text{liq}},x)}, \frac{1}{\theta}\right].$$

These are showed in Figure 3. Note that the lower the liquidity the bigger the slippage $\Delta(P_{\text{liq}},x)$ yielding bigger (unrealistic) lower bounds for the liquidation bonus.

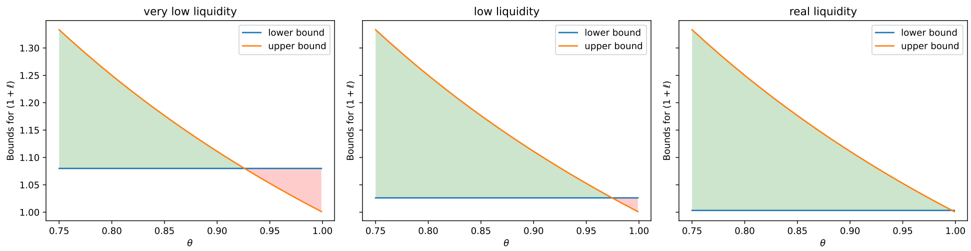

FIGURE 3. Bounds for the liquidation bonus under different liquidity scenarios. The red region covers the combination of $\theta,(1+\ell)$ values for which liquidation is not profitable

We extend the study by calculating the health factor after liquidation, $\mathbb{HF}_{t+}$, at each point of the $\theta,(1+\ell)$-grid for the different liquidity scenarios, see Figure 4. The red line provides the contour line where $\mathbb{HF}_{t+}=1$, and it divides the plot in the region where liquidation will trigger further liquidations ($\mathbb{HF}_{t+}<1$) and the region where liquidation will bring the position to a healthy factor ($\mathbb{HF}_{t+}>1$). We can see that for high values of θ, the liquidation bonus has to be low in order for the liquidation to be effective. As the market liquidity decreases, θ needs to be conservatively low to avoid liquidation spirals.

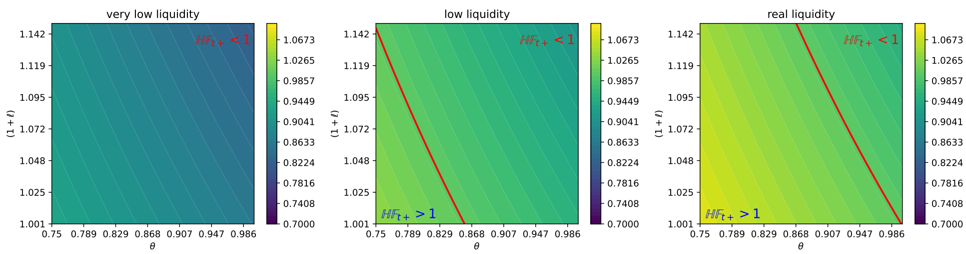

FIGURE 4. Contour plot depicting the health factor after liquidation, $\mathbb{HF}_{t+}$ (cf. (5)) for the three most liquid scenarios considered in Table 1.

We can use Figure 4 to correct the bounds depicted in Figure 3 so that the liquidation bonus is profitable for the liquidator AND does not cause further liquidations on the same position. We superimpose both figures yielding the updated bounds in Figure 5. Note that In the *very low liquidity* scenario and for the considered values of $\theta,(1+\ell)$ all liquidations will trigger further liquidations.

7.2.2. *Paradox of adversarial liquidation protocols.* We visualise the paradox of adversarial protocols using the scenarios from Figure 3. Recall that Figure 3 is created assuming that for each $\theta \in [0.7,1)$ the price of the collateral asset is such that loan is in distress. In Figure 6 we add the additional lower bound (in green) for $(1+\ell)$ such that running a front-run trade to trigger liquidation is profitable for an adversarial

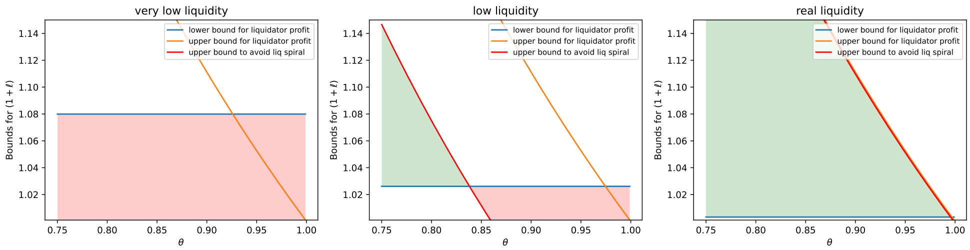

FIGURE 5. Bounds on liquidation bonus needed to avoid further liquidations

liquidator. We see that as market is less liquid the adversarial liquidation becomes more profitable.

Note that in the (fictitious) low liquidity scenarios, the price impact H of the front-run trade offset the slippage Δ of the front-run trade, and it would already be enough to have negative liquidation bonus for the procedure to be profitable.

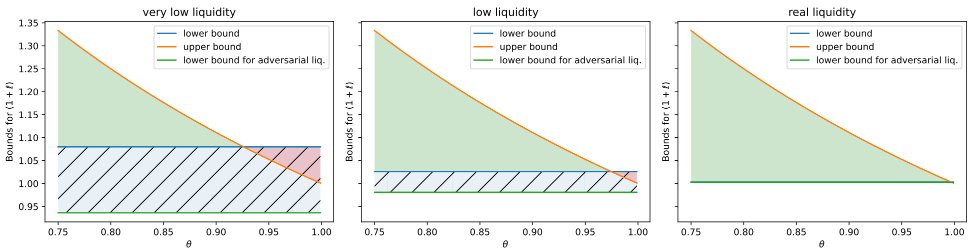

FIGURE 6. Paradox of of adversarial liquidation protocols. When $(1+\ell)$ is above blue line, the liquidation is profitable (see Lemmma 3). When $(1+\ell)$ is above green line the adversarial liquidation is profitable.

7.2.3. *Implied bound on the health factor.* We find the implied bound on the health factor in the three scenarios in Table 1, for different prices $P \in [1200, 1800]$. We remind the reader that following the paradox of adversarial liquidation protocols, if the liquidation bonus is set so that liquidation is effective, then adversarial liquidation is also effective. That is, in order to avoid adversarial liquidation it is not enough for the health factor to be greater than one, but it has to be greater than the bound

$$\left(1 - \frac{H(P, X(b,P))}{P}\right)^{-1}$$

where $X(b,p)$ is defined as the maximal solution to $x = \kappa b/(p - \Delta(p,x))$.

Figure 7 provides the Implied Bound depending on the price of ETH in USD in the three different liquidity scenarios from Table 1. As expected, the lower the liquidity, the higher the price impact $H(P, X(b,P))$ which in turn increases the implied bound.

7.2.4. *Cost to trigger liquidation.* In this section we plot the cost in WETH (the collateral asset) to trigger a liquidation. That is x such that

$$\mathbb{HF}_t \cdot \left(1 - \frac{H(P,x)}{P}\right) = 1,$$

in the three scenarios in Table 1. The cost is displayed in Figure 8. We see a positive correlation between the cost to trigger the liquidation and the liquidity of the external market.

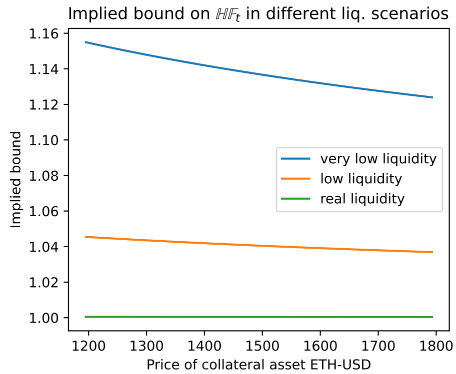

FIGURE 7. Implied bound on Health Factor

FIGURE 8. Cost to trigger liquidation

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APPENDIX A. PROOF OF PARADOX OF ADVERSARIAL LIQUIDATION PROTOCOLS

In this section we prove the inequalities leading to the paradox of adversarial liquidation protocols, namely

$$1+\ell > \frac{P_t - H(P_t,x)}{(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')} > \frac{P_t - H(P_t,x)}{P_t - \Delta(P_t,x)},$$

where x, x' are the consecutive trades that buy κb, solving

$$\begin{aligned}\kappa b &= x\,(P_t - \Delta(P_t,x))\\ \kappa b &= x'\left((P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')\right).\end{aligned}$$

The first inequality is provided by the bounds derived for the liquidation bonus so that liquidation procedure is profitable. In order to prove the second inequality, we need the auxiliary Lemma 5 that shows that $x' \geq x$, and then Lemma 2 proves the inequality.

Lemma 5. *Consider a CFM with convex trading function ψ, reserves X_t, Y_t, and consider the two consecutive trades*

(1) The trader buys κb units of the second asset and pays x units of the first asset.
reserves in the first asset move from X_t to $X_t + x$.
reserves in the second asset move from Y_t to $Y_t - \kappa b$
(2) The trader buys κb units of the second asset and pays x' units of the first asset.
reserves in the first asset move from $X_t + x$ to $X_t + x + x'$.
reserves in the second asset move from $Y_t - \kappa b$ to $Y_t - 2\cdot\kappa b$

Then, $x' \geq x$.

Proof. Each trade yields the following changes in the pool reserves

$$\begin{aligned}\psi(X_t) - \psi(X_t + x) &= \kappa b\\ \psi(X_t + x) - \psi(X_t + x + x') &= \kappa b,\end{aligned}$$

therefore

$$2\psi(X_t + x) - \psi(X_t) - \psi(X_t + x + x') = 0.$$

After rearranging and using the convexity of ψ, we get

$$\psi(X_t + x) = \frac{1}{2}\psi(X_t) + \frac{1}{2}\psi(X_t + x + x') \geq \psi\left(X_t + \frac{x+x'}{2}\right).$$

Since ψ is a decreasing function (higher reserves in one asset yield lower reserves in the second asset), we get

$$X_t + x \leq X_t + \frac{x+x'}{2} \Leftrightarrow x' \geq x.$$

□

Theorem 2. *In the CFM with level function $\Psi(x,y) = k$, and convex trading function $y = \psi(x)$ we have*

$$\frac{P_t - H(P_t,x)}{(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')} > \frac{P_t - H(P_t,x)}{P_t - \Delta(P_t,x)},$$

where x,x' are the consecutive trades that buy κb, solving

$$\begin{aligned}
\kappa b &= x\,(P_t - \Delta(P_t,x)) \\
\kappa b &= x'\left((P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x')\right).
\end{aligned}$$

Proof. It is enough to check

$$(P_t - H(P_t,x)) - \Delta(P_t - H(P_t,x),x') < P_t - \Delta(P_t,x),$$

i.e.

$$H(P_t,x) > \Delta(P_t,x) - \Delta(P_t - H(P_t,x),x'). \tag{12}$$

In a CFM we know the forms of $H(P_t,x),\Delta(P_t,x)$. Let X_t be the reserves amount of the collateral asset in the CFM at time t, then

$$\begin{aligned}
P_t &= -\psi'(X_t) \\
H(P_t,x) &= -\psi'(X_t) - (-\psi'(X_t+x)) \\
\Delta(P_t,x) &= -\psi'(X_t) - \left(-\frac{\psi(X_t+x) - \psi(X_t)}{x}\right).
\end{aligned}$$

Therefore, after replacing in (12), we get

$$\begin{aligned}
-\psi'(X_t) + \psi'(X_t+x) &> -\psi(X_t) + \frac{\psi(X_t+x) - \psi(X_t)}{x} \\
&\quad - \left(-\psi'(X_t+x) + \frac{\psi(X_t+x+x') - \psi(X_t+x)}{x'}\right) \\
\Leftrightarrow 0 &> \underbrace{\frac{\psi(X_t+x) - \psi(X_t)}{x}}_{=-\frac{\kappa b}{x}} - \underbrace{\frac{\psi(X_t+x+x') - \psi(X_t+x)}{x'}}_{=-\frac{\kappa b}{x'}} \\
\Leftrightarrow \frac{\kappa b}{x} - \frac{\kappa b}{x'} &> 0
\end{aligned}$$

which holds because from Lemma 5 we know that $x' > x$. □

APPENDIX B. DATA

We consider Uniswap v2, where $\Delta(P_t,x),H(P_t,x)$ have explicit form provided in Example 1. For simplicity, we assume zero-gas fees.

We query the Uniswap v2 GraphQL database [11] to obtain the liquidity constant k on the 1st of January of 2023. Furthermore, we create additional fictitious Uniswap pools with *lower liquidity* $k_{-n} = 10^{-n}k$ for $n = 4,5$.

Table 1 provides the Uniswap v2 liquidity scenarios.

[11] `https://api.thegraph.com/subgraphs/name/ianlapham/governance-tracking`

Uni. v2 liquidity	Uni. v2 reserves	Label in plots
$10^{-5}k$	WETH: 30.91, USDT: 35114.06	Very low liquidity
$10^{-4}k$	WETH: 97.75, USDT: 111040.40	Low liquidity
k	WETH: 9775.18, USDT: 11104040.38	Real liquidity

TABLE 1. Uniswap v2 liquidity scenarios, with the last row being the real one on the 1st of January queried from the Graph, and $k = 108544044156.00478$. The third column provides the labels we use in Figures 3, 4, 5, 7, 8.